TOWARDS $Sub(\mathbb{Z}^N)$ IMPLEMENTATION OF PROTEIN-PROTEIN INTERACTIONS

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Contents

1. Introduction

Let $PROT$ be the collection of all proteins and their complexes that are encoded by the genes of an organism. And let $INTR$ be the collection of all interactions among them. We regard $INTR$ as actions on $Sub(PROT)$: $INTR \times Sub(PROT) \to Sub(PROT)$,

$$(interaction, \{protein_{11}, \ldots, protein_{1m}\}) \mapsto \{protein_{21}, \ldots, protein_{2n}\}.$$

where we write $Sub(XX)$ for the collection of all finite subsets of XX. And we propose a new system $\mathbb{HN}^N$ ($N \in \mathbb{N}$), called N-dimensional *hetero numbers*, to implement the action of $INRT$ on $Sub(PROT)$. Then, by assigning hetero numbers to proteins and their complexes, protein-protein interactions are implemented as an action of a finite subset of $\mathbb{Z}^N$ on $Sub(\mathbb{HN}^N)$.

Recall that we use weights in scales for weighing objects and a ruler to measure their length. $\mathbb{HN}^N$ is a system of units for measuring shapes of objects such as proteins ([1]). It is characterized by correspondences between

(1) protein-protein interaction and addition (Fig.1(a)),

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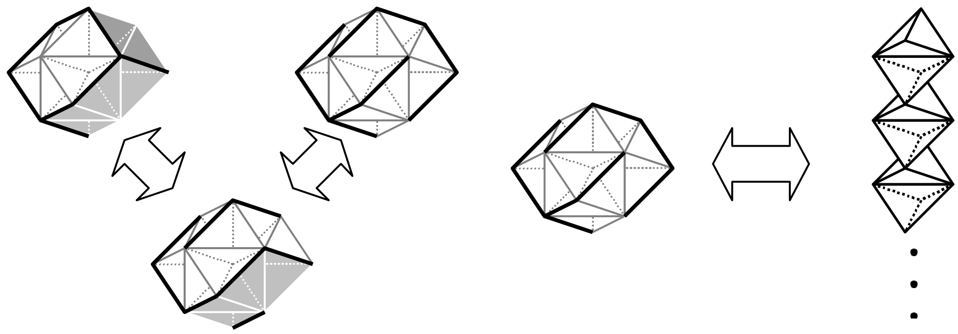


(a) (b)

FIGURE 1. Features of hetero numbers.

(2) genetic code and the second derivative (Fig.1(b)).

And $\mathbb{HN}^N$ gives an example of commutative higher-dimensional extension of $\mathbb{N}$. In particular, we obtain "genetic codes" of natural numbers via a natural embedding of $\mathbb{N}$ into $\mathbb{PHN}^2$.

1.1. **Simultaneous equations.** In this paper, we shall consider a kind of simultaneous equations. Let Z and A be finite sets of indeterminates. We assign a finite subset of Z to each $a \in A$ and define an "action" $*$ of a on it. That is,

$$a * \{x_1, x_2, \ldots, x_m\} := \{y_1, y_2, \ldots, y_n\}$$

for some $\{x_1, x_2, \ldots, x_m\}$ and $\{y_1, y_2 \ldots, y_n\} \in Sub(Z)$. $\{x_1, x_2, \ldots, x_m\}$ is called the *domain* of a and written *domain*(a). $\{y_1, y_2, \ldots, y_n\}$ is called the *codomain* of a and written *codom*(a). Moreover, we impose "conservation law" of some quantity ν for each "action" of $a \in A$:

$$\nu(x_1) + \nu(x_2) + \cdots + \nu(x_m) = \nu(y_1) + \nu(y_2) + \cdots + \nu(y_n).$$

Example 1.1. $Z = \{z1, z2, z3, z4\}$, $A = \{a\}$, and

$$a * \{z1, z2, z3\} := \{z4\}.$$

Example 1.2. $Z = \{z1, z2, z3, z4, z5, z6\}$, $A = \{a1, a2\}$, and

$$a1 * \{z5\} := \{z1, z2, z3\},$$
$$a2 * \{z2, z3, z4\} := \{z6\}.$$

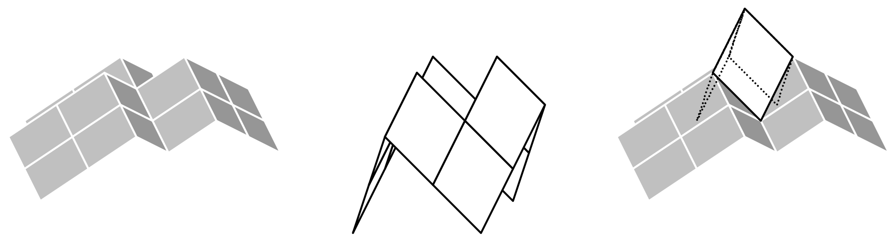
Example 1.3. $Z = \{z1, z2, z3, z4, z5, z6, z7\}$, $A = \{a1, a2, a3\}$, and

$$a1 * \{z1, z2\} := \{z5\},$$
$$a2 * \{z3, z4\} := \{z6\},$$
$$a3 * \{z5, z6\} := \{z7\}.$$

Fig.1(a) shows $\{z1, z2, z3, z4\} \Leftrightarrow \{z5, z6\} \Leftrightarrow \{z7\}$ of a solution of example 1.3. See example 3.4 for more details.

Note that A defines a functor I with domain $Sub(Z)$ and codomain $Sub(A)$:

$$I(U) := \{a \in A \mid domain(a) \subset U\}.$$



(a) (b) (c)

FIGURE 2. 3-dim. Cones and roofs. (a): $Cone^*\{x_1x_2, x_2x_3, x_1x_3\}$. (b): $Cone\{y_1, y_2, y_3\}$. (c): $Cone^*\{x_1x_2, x_2x_3, x_1x_3\}$ and $Roof\{y_1, y_2, y_3\}$. Note that $y_1 = x_2x_3, y_2 = x_1x_3$ and $y_3 = x_1x_2$.

In the following, we shall construct a "global section" of I by patching "local sections" $a \in A$ together to solve the simultaneous equations. I is called the *interaction functor* associated with the equations.

1.2. **Biological background.** A protein is a polymer of amino-acids linked by peptide bonds, the order of whose amino-acids are encoded by a gene. In nature, each protein is folded into a well-defined three-dimensional ($3D$) structure. And the functional properties of a protein depend on its $3D$ structure. For example, the $3D$ structure determines the active site of an enzyme, the binding site of a drug, or binding site for another protein. That is, $3D$ structure is the key to characterize the biological function of proteins.

In the case of protein-protein interactions, proteins usually form transient or stable complexes to do their jobs. And protein complexes are classified manually by structural similarity.

2. Hetero numbers

2.1. **Preparation.** For $N \in \mathbb{N}$, we define two N-dimensional lattices.

Definition 2.1 (L_N and its conjugate lattice ${L_N}^*$)**.**

$$L_N := \left\{ {y_1}^{l1} {y_2}^{l2} \cdots {y_N}^{lN} \mid l_i \in \mathbb{Z} \text{ for all } i \right\},$$
$${L_N}^* := \left\{ {x_1}^{l1} {x_2}^{l2} \cdots {x_N}^{lN} \mid l_i \in \mathbb{Z} \text{ for all } i \right\},$$

where $y_k = x_1x_2 \cdots x_N/x_k$ $(1 \le k \le N)$. In particular, ${L_N}^*/L_N \sim \left\{1, x_1, {x_1}^2, \ldots, {x_1}^{N-2}\right\}$, where $a \equiv b \mod (L_N)$ if and only if $a/b \in L_N$. Note $x_i \equiv x_j \mod (L_N)$.

Two types of cones are defined for each lattice.

Definition 2.2 (Cones of L_N)**.** For $A \subset L_N$,

$$Cone\ A := \left\{ p{y_1}^{l1} {y_2}^{l2} \cdots {y_N}^{lN} \in L_N \mid p \in A \text{ and } 0 \le l_i \in \mathbb{Z} \text{ for all } i \right\},$$
$$Roof\ A := \{ p \in L_N \mid \exists m \in \mathbb{N} \text{ s.t. } p{y_i}^m \in Cone\ A \text{ for all } i \}.$$

Note that $Cone\ A \subset Roof\ A$ for any $A \subset L_N$. See Fig.2.

We denote the "peaks" of cone w of L_N by $g(w)$. That is, $g(w) \subset L_N$ is the minimal system of elements of w which satisfies $w = Cone\ g(w)$. In particular, $Roof\ A = Cone\ g(Roof A)$ for any $A \subset L_N$. And we denote the "boundary surface" of cone (or roof) w of L_N by dw. That is, $dw := \{z \in w \mid l_w(z) \le 0\}$, where $l_w(z) := \max_{p \in g(w)} \left\{ \min_{1 \le i \le N} \left\{ l_i \mid \prod_{1 \le i \le N} {y_i}^{li} = z/p \right\} \right\}$.

Cones of ${L_N}^*$ are also defined similarly.

Definition 2.3 (Cones of ${L_N}^*$)**.** For $A \subset {L_N}^*$, we define

$$Cone^* A := \left\{ p{x_1}^{l1}{x_2}^{l2} \cdots {x_N}^{lN} \in {L_N}^* \mid p \in A \text{ and } 0 \le l_i \in \mathbb{Z} \text{ for all } i \right\},$$
$$Roof^* A := \left\{ p \in {L_N}^* \mid \exists m \in \mathbb{N} \text{ s.t. } p{x_i}^m \in Cone^* A \text{ for all } i \right\}.$$

2.2. **Hetero numbers.** Now we define the category of hetero numbers.

Definition 2.4 (Prehetero numbers and hetero numbers)**.**

$$\mathbb{PHN}^N := \{Cone^* A \mid A \subset {L_N}^*\},$$
$$\mathbb{HN}^N := \{Roof^* A \mid A \subset L_N\},$$
$$X := \{Roof\ A \mid A \subset L_N\}.$$

$\mathbb{PHN}^N$ forms the category of N-dimensional *prehetero numbers* with arrows $Cone^* A \to Cone^* B$ the inclusions $Cone^* A \subset Cone^* B$. $\mathbb{HN}^N$ forms the category of N-dimensional *prehetero numbers* with the inclusions similarly. X also forms a category with arrows the inclusions $Roof\ A \subset Roof\ B$. And there are natural projections $\nu : \mathbb{PHN}^N \to \mathbb{HN}^N$ and $dom : \mathbb{PHN}^N \to X$ defined by

$$\nu(w) := Roof^* g(Cone^*(w \cap L_N)),$$
$$dom(w) := Roof\ g(Cone^*(w \cap L_N)).$$

We identify $\mathbb{PHN}^N$ and $\mathbb{HN}^N$ with functions from X to $\mathbb{PHN}^N$ defined by

$$w(q) := Cone^*(g(w) \cap q) \quad \text{for } w \in \mathbb{PHN}^N,$$
$$v(q) := Cone^*(dv \cap q) \quad \text{for } v \in \mathbb{HN}^N.$$

Note that $w(dom(w)) = w$ for $w = Cone^* B \in \mathbb{PHN}^N$ when $B \subset L_N$.

Algebraic structures are defined as follows.

Definition 2.5 (Algebra of $\mathbb{HN}^N$ and X)**.**

$$\text{[Addition]} \quad Roof^* A + Roof^* B := Roof^*(A \cup B),$$
$$\text{[Multiplication]} \quad Roof^* A\ Roof^* B := Roof^* AB,$$
$$\text{[Scalar Multiplication]} \quad A\ Roof^* B := Roof^* AB,$$

where $A, B \subset L_N$ and $AB := \{pq \mid p \in A \text{ and } q \in B\}$.

Algebra of X is also defined similarly.

Definition 2.6 (Algebra of $\mathbb{PHN}^N$)**.**

$$\text{[Addition]} \quad Cone^* A \cup Cone^* B := Cone^*(A \cup B),$$
$$\text{[Multiplication]} \quad Cone^* A\ Cone^* B := Cone^* AB,$$
$$\text{[Scalar Multiplication]} \quad A\ Cone^* B := Cone^* AB,$$

where $A, B \subset {L_N}^*$.

We also define an action of $Sub({L_N}^*)$ on $\mathbb{PHN}^N$ by

$$* : Sub({L_N}^*) \times \mathbb{PHN}^N \to \mathbb{PHN}^N,\ A * Cone^* B := Cone^*(A \cup B).$$

Note that it induces an action of $Sub(\mathbb{Z}^N)$ on $\mathbb{PHN}^N$ since ${L_N}^* \sim \mathbb{Z}^N$.

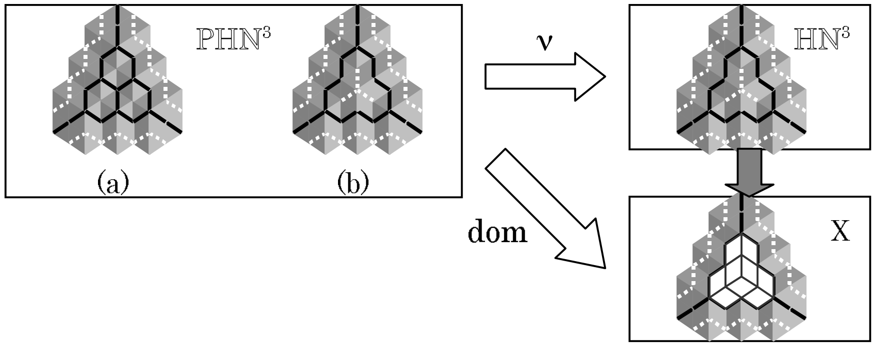


FIGURE 3. Atomic decompositions of $Roof^*A^2$ over $Roof\ \{1\}$, where $A = \{x_1x_2, x_2x_3, x_1x_3\}$. (a): $Cone^*A^2 = Cone^*\{x_1x_2\}A \cup Cone^*\{x_2x_3\}A \cup Cone^*\{x_1x_3\}A$. (b): $\{x_1x_2x_3\} * Cone^*A^2$.

2.3. **Atomic decompositions of hetero numbers.** For $w \in \mathbb{PHN}^N$, set

$$M(w) := \{Roof\ A \mid A \subset g(w) \cap L_N,\ g(Roof\ A) \cap g(Cone\ A) = \emptyset\} \subset X.$$

w is called *degenerated* if $M(w) = \emptyset$ and called *atomic* if $M(w)$ consists of a single element.

Proposition 2.7 (Atomic decomposition of $w \in \mathbb{PHN}^N$)**.** *Suppose that $dom(w) = \sum_{q \in M(w)} q$ and $w(dom(w)) = w$. Then, w is decomposed into atomics uniquely:*

$$\exists! w_i \in \mathbb{PHN}^N\ (1 \leq i \leq k)\ s.t.\ \begin{cases} w = \bigcup_{1 \leq i \leq k} w_i, \\ w_i \text{ is atomic for } \forall i. \end{cases}$$

In particular, $\nu(w) = \sum_{1 \leq i \leq k} \nu(w_i)$ and we call it an atomic decomposition of $\nu(w) \in \mathbb{HN}^N$ over $dom(w) \in X$.

Proof. Let $\{q_i \mid 1 \leq i \leq k\}$ be the collection of all minimal elements of $M(w)$. Then, $w = w(dom(w)) = \bigcup_i w(q_i)$ because $dom(w) = \sum_i q_i$. We obtain the result by setting $w_i = w(q_i)$ $(1 \leq i \leq k)$. □

Remark 2.8. As we shall see in the next section, w corresponds to a collection of loops of simplices. And we also obtain the result immediately by describing the loops algebraically.

For $v \in \mathbb{HN}^N$ and $q \in X$, set

$$W(v,q) := \nu^{-1}(v) \cap dom^{-1}(q) \ \subset \mathbb{PHN}^N,$$
$$I(v,q) := \{p \in {L_N}^* \mid \{p\} * W(v,q) \subset W(v,q)\} \ \subset {L_N}^*.$$

$W(v,q)$ corresponds to the collection of all atomic decompositions of v over q. And $p \in I(v,q)$ induces a transition between them. For example, as you see in Fig.3, $Roof^*\{x_1x_2, x_2x_3, x_1x_3\}^2$ has two atomic decompositions (a) and (b) over $Roof\ \{1\}$. And $x_1x_2x_3 \in L_N^*$ induces a transition between them. Note that $W(v,q)$ is uniquely determined by $v(q) \in \mathbb{PHN}^N$ if $v = \nu(v(q))$ and $q = dom(v(q))$.

In the following, we identify a decomposition $\bigcup_i w_i \in \mathbb{PHN}^N$ with a subset $\{\nu(w_i)\}$ of $\mathbb{HN}^N$ to implement an interaction functor I in $I(v,q)$ for some v and q.

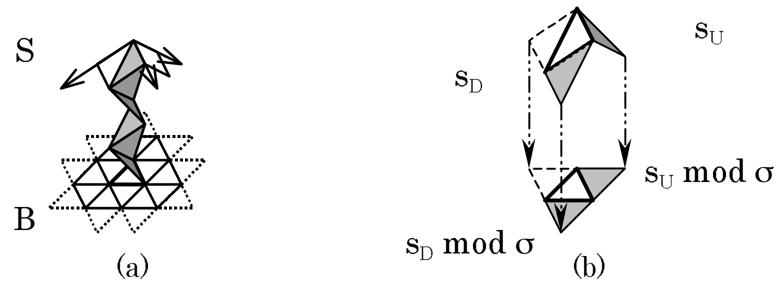


FIGURE 4. Differential geometry of 2-simplices. (a): Fiber of S over a point of B. (b): The local orbit defined by $s \in S$.

2.4. **Differential geometry of hetero numbers.** We associate lattice ${L_N}^*$ with a collection S of $N-1$-simplices:

$$S := \left\{ a \left[x_{\rho(1)} \cdots x_{\rho(N-1)} \right] \mid a \in {L_N}^*,\ \rho \in S_N \right\},$$

where S_N is the N-th symmetric group and $a \left[x_{\rho(1)} \cdots x_{\rho(N-1)} \right]$ denotes the convex hull of N points $a_0 = a, a_1 = ax_{\rho(1)}, \ldots, a_{N-1} = ax_{\rho(1)}x_{\rho(2)} \cdots x_{\rho(N-1)}$ in $\mathbb{R}^N$:

$$a \left[x_{\rho(1)} \cdots x_{\rho(N-1)} \right] := \left\{ \prod_{0 \le i < N} a_i^{\lambda_i} \mid 0 \le \lambda_i \in \mathbb{R} \text{ s.t. } \sum_{0 \le i < N} \lambda_i = 1 \right\}.$$

Using a "shift operator" $\sigma : S \to S$, $\sigma \left(a \left[x_{\rho(1)} \cdots x_{\rho(N-1)} \right] \right) := ax_{\rho(1)} \left[x_{\rho(2)} \cdots x_{\rho(N)} \right]$, we define a "tangent bundle" $T[B] := S/\sigma^N$ on $B := S/\sigma$ (Fig.4(a)):

$$\pi : T[B] \to B,\ \pi \left(s \mod \sigma^N \right) := s \mod \sigma.$$

Note that $T[B] \sim B \times \{e/x_1, e/x_2, \ldots, e/x_N\}$ ($e = x_1 x_2 \cdots x_N$) by $s \mod \sigma^N \sim (s \mod \sigma, Ds)$, where

$$Da \left[x_{\rho(1)} \cdots x_{\rho(N-1)} \right] := x_{\rho(1)} \cdots x_{\rho(N-1)} = e/x_{\rho(N)}.$$

Ds is called the *gradient* of $s \in S$. A "tangent vector" $s \mod \sigma^N \in T[B]$ is associated with a *local trajectory* $\{s_U \mod \sigma, s \mod \sigma,\ s_D \mod \sigma\}$ at $s \mod \sigma$ (Fig.4(b)), where $s = a \left[x_{\rho(1)} \cdots x_{\rho(N-1)} \right]$,

$$s_U := a \left[x_{\rho(1)} \cdots x_{\rho(N-2)} x_{\rho(N)} \right] \text{ and } s_D := ax_{\rho(1)} \left[x_{\rho(2)} \cdots x_{\rho(N-1)} x_{\rho(N)} \right].$$

Then, $w \in \mathbb{PHN}^N$ induces a vector field on B naturally:

$$X_w : B \to \{e/x_1, e/x_2, \ldots, e/x_N\},\ X_w(s \mod \sigma) := D\Gamma_w(s \mod \sigma),$$

where $\Gamma_w(s \mod \sigma) := d_{\mathbb{R}} w \cap O_{s \mod \sigma}$, $d_{\mathbb{R}} w := \{s \in S \mid$ all vertices of s are contained in $dw\}$, and $O_{s \mod \sigma} := \{\sigma^n(s) \mid n \in \mathbb{Z}\}$. We associate w with the collection of all closed trajectories, or simply *loops*, defined by X_w. And we identify two prehetero numbers w_1 and w_2 if they correspond to the same collection of loops. For example, in Fig.3, $Cone^* A^2$ consists of three loops of length 6 and $\{x_1 x_2 x_3\} * Cone^* A^2$ consists of one loop of length 18. See appendix B for the distribution of loops.

Note that $w \in \mathbb{PHN}^3$ is atomic if and only if it corresponds to one loop. But, if $N > 3$, some atomic prehetero numbers consist of several loops and some loops are

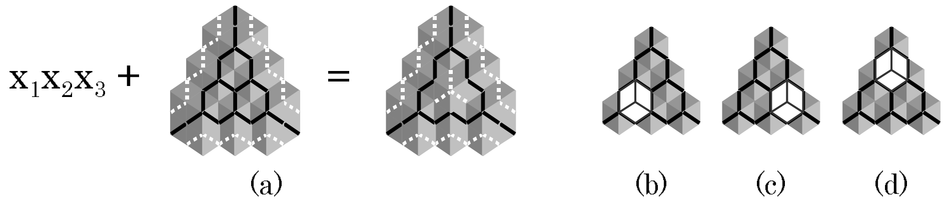


FIGURE 5. A solution of example 1.1.

degenerated algebraically. In other words, loop decompositions give refinements of atomic decompositions. See example 3.4.

3. $Sub(\mathbb{Z}^N)$ IMPLEMENTATION OF ADDITION OF $\mathbb{HN}^N$

3.1. **Solutions of simultaneous equations in $\mathbb{PHN}^N$.** Now we shall solve the simultaneous equations of $Z = \{z_1, \dots, z_m\}$ and $A = \{a_1, \dots, a_n\}$ in $\mathbb{PHN}^N$ by embedding the interactions A in $I(v, q)$ for some $v \in \mathbb{HN}^N$ and $q \in X$. That is, we shall find $\{w_1, \dots, w_m\} \subset \mathbb{PHN}^N$ and $\{p_1, \dots, p_n\} \subset {L_N}^*$ which satisfy

$$\begin{cases} \{p_i\} * \left(\bigcup_{\{j|z_j \in domain(a_i)\}} w_j\right) = \bigcup_{\{j|z_j \in codom(a_i)\}} w_j & \text{in } \mathbb{PHN}^N, \\ \sum_{\{j|z_j \in domain(a_i)\}} dom(w_j) = \sum_{\{j|z_j \in codom(a_i)\}} dom(w_j) & \text{in } X, \end{cases}$$

and a conservation law

$$\sum_{\{j|z_j \in domain(a_i)\}} \nu(w_j) = \sum_{\{j|z_j \in codom(a_i)\}} \nu(w_j) \qquad \text{in } \mathbb{HN}^N.$$

Note that $v = \sum_{1 \le i \le m} \nu(w_i)$ and $x = \sum_{1 \le i \le m} dom(w_i)$.

Then, the interaction functor I corresponds to a functor, also denoted by I, from $Sub(X)$ to $Sub({L_N}^*)$: for $U \subset X$,

$$p_i \in I(U) \text{ if and only if } \{dom(w_j) | z_j \in domain(a_i)\} \subset U.$$

Remark 3.1. Geometrically speaking, $I(v, q)$ preserves the region swept by the loops corresponds to $w \in W(v, q)$. Thus, the simultaneous equations give the defining equations of the swept region if $A = I(v, q)$. In other words, the "shape" of the region gives the "semantics" of the interactions A.

Example 3.2 (Solution of example 1.1 in $\mathbb{PHN}^3$ (Fig.5))**.** Set

$$\begin{aligned} w_1 &= Cone^*\{x_2x_3\}\{x_1x_2, x_2x_3, x_3x_1\} \quad \text{(Fig.5(b))}, \\ w_2 &= Cone^*\{x_3x_1\}\{x_1x_2, x_2x_3, x_3x_1\} \quad \text{(Fig.5(c))}, \\ w_3 &= Cone^*\{x_1x_2\}\{x_1x_2, x_2x_3, x_3x_1\} \quad \text{(Fig.5(d))}, \\ w_4 &= Cone^*\{{x_1}^2{x_2}^2, {x_2}^2{x_3}^2, {x_3}^2{x_1}^2, x_1x_2x_3\}. \end{aligned}$$

Then we have

$$\begin{cases} \{x_1x_2x_3\} * (w_1 \cup w_2 \cup w_3) = w_4, \\ dom(w_1) + dom(w_2) + dom(w_3) = dom(w_4), \\ \nu(w_1) + \nu(w_2) + \nu(w_3) = \nu(w_4). \end{cases}$$

Example 3.3 (Solution of example 1.2 in $\mathbb{PHN}^3$)**.** Set

$$\begin{aligned}
w_1 &= Cone^*\{{x_2}^2\}\{x_1x_2, x_2x_3, x_3x_1\},\\
w_2 &= Cone^*\{x_1x_2\}\{x_1x_2, x_2x_3, x_3x_1\},\\
w_3 &= Cone^*\{x_2x_3\}\{x_1x_2, x_2x_3, x_3x_1\},\\
w_4 &= Cone^*\{x_3x_1\}\{x_1x_2, x_2x_3, x_3x_1\},\\
w_5 &= Cone^*\{{x_1}^2{x_2}^2, {x_1}^2x_2x_3, x_1x_2{x_3}^2, {x_2}^2{x_3}^2, {x_2}^3x_3, x_1{x_2}^3\},\\
w_6 &= Cone^*\{{x_1}^2{x_2}^2, {x_2}^2{x_3}^2, {x_3}^2{x_1}^2, x_1x_2x_3\}.
\end{aligned}$$

Then we have

$$\begin{cases}
\{x_1{x_2}^2x_3\} * w_5 = w_1 \cup w_2 \cup w_3,\\
dom(w_5) = dom(w_1) + dom(w_2) + dom(w_3),\\
\nu(w_5) = \nu(w_1) + \nu(w_2) + \nu(w_3),
\end{cases}$$

$$\begin{cases}
\{x_1x_2x_3\} * (w_2 \cup w_3 \cup w_4) = w_6,\\
dom(w_2) + dom(w_3) + dom(w_4) = dom(w_6),\\
\nu(w_2) + \nu(w_3) + \nu(w_4) = \nu(w_6).
\end{cases}$$

As for example 1.3, the “fusion and fission” of a rhombic dodecahedron in Fig.1(a) gives a solution in $\mathbb{HN}^4$.

Example 3.4 (Solution of example 1.3 in loops of $\mathbb{PHN}^4$)**.** Set

$$\begin{aligned}
lp_1 &= Cone^*\{x_1x_3x_4, x_1x_2x_4, x_1x_2x_3\},\\
lp_2 &= Cone^*\{x_2x_3x_4, x_1x_3x_4, x_1x_2x_3\},\\
lp_3 &= Cone^*\{x_2x_3x_4, x_1x_2x_4, x_1x_2x_3\},\\
lp_4 &= Cone^*\{x_2x_3x_4, x_1x_3x_4, x_1x_2x_4\},\\
lp_5 \cup lp_3 \cup lp_4 &= Cone^*\{x_1x_2x_4, x_2x_3x_4, x_1x_3\},\\
lp_6 \cup lp_1 \cup lp_2 &= Cone^*\{x_1x_2x_3, x_1x_3x_4, x_2x_4\},\\
lp_5 \cup lp_6 &= Cone^*\{x_1x_3, x_2x_4\},\\
lp_7 &= Cone^*\{x_1x_3, x_2x_3, x_2x_4\}.
\end{aligned}$$

Then lp_1, lp_2, lp_3 and lp_4 are loops of length 6, lp_5 and lp_6 are loops of length 12, and lp_7 is a loop of length 24 (Fig.1(b)). And they satisfy

$$\begin{cases}
\{x_1x_3\} * (lp_1 \cup lp_2 \cup lp_3 \cup lp_4) = lp_5 \cup lp_3 \cup lp_4,\\
dom(lp_1) + dom(lp_2) + dom(lp_3) + dom(lp_4) = dom(lp_5 \cup lp_3 \cup lp_4),\\
\nu(lp_1) + \nu(lp_2) + \nu(lp_3) + \nu(lp_4) = \nu(lp_5 \cup lp_3 \cup lp_4),
\end{cases}$$

$$\begin{cases}
\{x_2x_4\} * (lp_1 \cup lp_2 \cup lp_3 \cup lp_4) = lp_6 \cup lp_1 \cup lp_2,\\
dom(lp_1) + dom(lp_2) + dom(lp_3) + dom(lp_4) = dom(lp_6 \cup lp_1 \cup lp_2),\\
\nu(lp_1) + \nu(lp_2) + \nu(lp_3) + \nu(lp_4) = \nu(lp_6 \cup lp_1 \cup lp_2),
\end{cases}$$

$$\begin{cases}
\{x_2x_3\} * (lp_5 \cup lp_6) = lp_7,\\
dom(lp_5 \cup lp_6) = dom(lp_7),\\
\nu(lp_5 \cup lp_6) = \nu(lp_7).
\end{cases}$$

In the above, we identified $Cone^*\{x_1x_2x_3, x_1x_2x_4, x_1x_3x_4, x_2x_3x_4\}$ with $Cone^*\{ x_1x_3, x_2x_3, x_2x_4\}\{x_1, x_2, x_3, x_4\}$ since they correspond to the same collection of

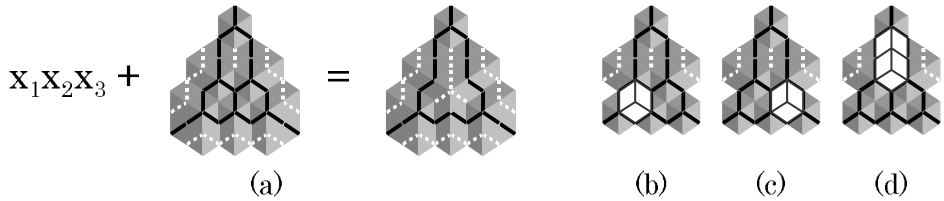


FIGURE 6. Another solution of example 1.1.

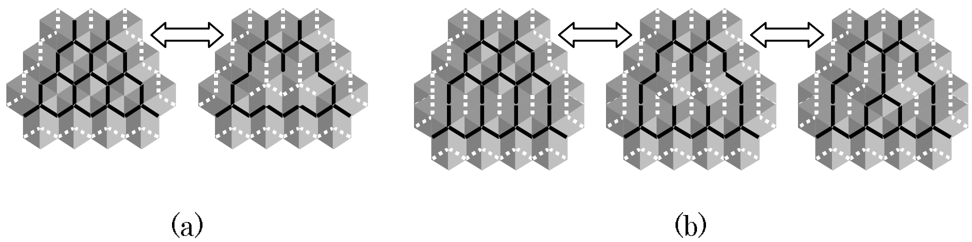


FIGURE 7. “External” interactions of example 1.1

loops. Note that $lp_1, \cdots, lp_4$ are degenerated. And we cannot divide each of $lp_5 \cup lp_3 \cup lp_4, lp_6 \cup lp_1 \cup lp_1$, and $lp_5 \cup lp_6$ algebraically.

3.2. **Internal interactions and external interactions.** So far, we have considered “internal” interactions. As a result, solutions are not determined uniquely. For example, the following example gives another solution of example 1.1.

Example 3.5 (Another solution in $\mathbb{PHN}^3$ of example 1.1 (Fig.6))**.** Set

$$
\begin{aligned}
w_1 &= Cone^*\{x_2x_3\}\{x_1x_2, x_2x_3, x_3x_1\} \quad \text{(Fig.6(b))},\\
w_2 &= Cone^*\{x_3x_1\}\{x_1x_2, x_2x_3, x_3x_1\} \quad \text{(Fig.6(c))},\\
w_3' &= Cone^*\{x_1x_2\}\{\mathbf{x_1}^{\mathbf{2}}\mathbf{x_2}^{\mathbf{2}}, x_2x_3, x_3x_1\} \quad \text{(Fig.6(d))},\\
w_4' &= Cone^*\{\mathbf{x_1}^{\mathbf{3}}\mathbf{x_2}^{\mathbf{3}}, {x_2}^2{x_3}^2, {x_3}^2{x_1}^2, x_1x_2x_3\}.
\end{aligned}
$$

Then we have

$$
\begin{cases}
\{x_1x_2x_3\} * (w_1 \cup w_2 \cup w_3') = w_4',\\
dom(w_1) + dom(w_2) + dom(w_3') = dom(w_4'),\\
\nu(w_1) + \nu(w_2) + \nu(w_3') = \nu(w_4').
\end{cases}
$$

But we can distinguish one from another by considering more variables as follows.

Example 3.6 (“External” interaction of example 1.1 (Fig.7))**.** Let

$$
\begin{aligned}
Z_E &= \{z11, z12, z13, z14, z15, z16, z17, z18, z19, z20\},\\
A_E &= \{a1, a2, a3, a4\}.
\end{aligned}
$$

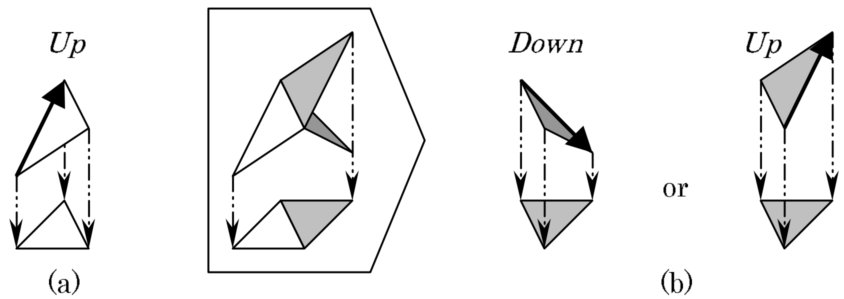


FIGURE 8. The second derivative of Γ_w along $\{t[i]\} \subset B$ ($N = 3$). (a): The current simplex $t[i]$. (b): The next simplex $t[i+1]$.

And define interactions A_E between $z3$ and Z_E by

$$\begin{aligned} a1 * \{z_3, z11, z12\} &= \{z15\}, \\ a2 * \{z_3, z13, z14\} &= \{z16\}, \\ a3 * \{z13, z14, z15\} &= \{z17\}, \\ a3 * \{z17\} &= \{z18, z19, z20\}. \end{aligned}$$

Then, on one hand, $z3$ of example 3.5 satisfies all four interactions. Fig.7(b) shows $\{z3, z11, z12, z13, z14\} \Leftrightarrow \{z17\} \Leftrightarrow \{z18, z19, z20\}$. On the other hand, $z3$ of example 1.1 satisfies only three interactions $a1$, $a2$, and $a3$. Fig.7(a) shows $\{z3, z11, z12, z13, z14\} \Leftrightarrow \{z17\}$.

In general, let U and V be disjoint subsets of Z, i.e. $U \cap V = \emptyset$, and write $I(U \cup V; U, V)$ for the quotient set $I(U \cup V)/(I(U) \cup I(V))$:

$$0 \longrightarrow I(U) \cup I(V) \longrightarrow I(U \cup V) \longrightarrow I(U \cup V; U, V) \longrightarrow 0.$$

We call $I(U \cup V; U, V)$ the *external interaction* between U and V induced by I. $I(U)$ is called the *internal interaction* of U induced by I.

Note that there arise two types of problems. That is,

(1) When is U determined uniquely by $I(U \cup V; U, V)$?
(2) What interactions $I(U \cup V; U, V)$ are possible between U and V?

APPENDIX A. "GENETIC CODE" OF HETERO NUMBERS

Here we shall consider the second derivative of Γ_w for $w \in \mathbb{PHN}^N$, which is a binary-valued function on B. Let $v \in \mathbb{HN}^N$, $q \in X$ and suppose that there exists $w \in W(v, q)$ s.t. w corresponds to a single loop. Then, the second derivative of w along the loop gives a binary code of v over q. We call it a *genetic code* of v over q.

Definition A.1 (The second derivative of Γ_w)**.** Let $\{t[i]\} \subset B$ be a trajectory of the vector field X_w of $w \in \mathbb{PHN}^N$. The second derivative of Γ_w along $\{t[i]\}$ is a $\{Up, Down\}$-valued function defined by

$$D^2\Gamma_w(t[i+1]) := \begin{cases} D^2\Gamma_w(t[i]) & \text{if } X_w(t[i+1]) = X_w(t[i]), \\ -D^2\Gamma_w(t[i]) & \text{else,} \end{cases}$$

where $-Down := Up$ and $-Up := Down$. See Fig. 8 for the case of $N = 3$.

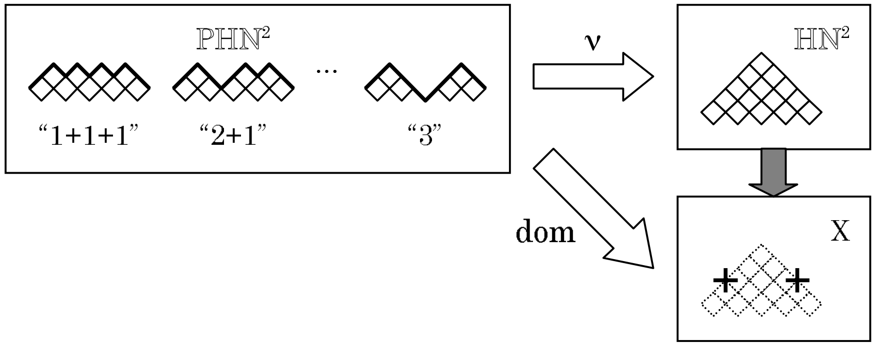


FIGURE 9. $W(v_3, q_3)$.

Example A.2 ("Genetic code" of a hexagon)**.** $Cone^*\{x_1x_2, x_2x_3, x_1x_3\} \in \mathbb{PHN}^3$ specifies a loop of length 6 (Fig.3). Since the loop sweeps a hexagon, Its second derivative along the loop gives a "genetic code" of a hexagon:

$$U - D - U - D - U - D.$$

Example A.3 ("Genetic code" of a rhombic dodecahedron)**.** $Cone^*\{x_1x_3, x_2x_3, x_2x_4\} \in \mathbb{PHN}^4$ specifies a loop of length 24 (Fig.1(b)). Then, its second derivative along the loop gives a "genetic code" of a rhombic dodecahedron:

$$\begin{aligned} U - D - D - U - D - U - U - D \\ - U - D - D - U - D - U - U - D \\ - U - D - D - U - D - U - U - D. \end{aligned}$$

Example A.4 ("Genetic code" of natural numbers)**.** Natural numbers give examples of 2-dimensional hetero numbers. Since ${L_2}^* = L_2$, we set

$$X := L_2 \times L_2,$$
$$dom(w) := (\text{the left-end peak of } w, \text{the right-end peak of } w).$$

And we define an embedding

$$\iota : \mathbb{N} \to X,\ \iota(k) := \{{x_1}^k, {x_2}^k\}.$$

Then, the genetic code of $v_k = Roof^*\ \{{x_1}^k, {x_2}^k\}$ over $q_k = \iota(k)$ gives the "genetic code" of $k \in N$. Note that $W(v_k, q_k)$ corresponds to the collection of all additive decomposition of k.

For example, $W(v_3, q_3) \Leftrightarrow \{$"$1+1+1$", "$2+1$", "$1+2$",$\ldots$, "3"$\}$ (Fig.9) and the second derivative of $\Gamma_{\text{“}3\text{”}}$ gives a "genetic code" of $3 \in \mathbb{N}$:

$$D - D - D - U - U - U.$$

Appendix B. Distribution of loops

Table 1. Distribution of loops. The table shows the distribution of the second derivative of loops. Loops are identified if they coincide with each other by translation, by rotation, by reflection, or by any combination of them. Note that values in the column of $3 \leq N \leq 8$ are not equal to the sum of values in the right columns since some loops share the same second derivative.

length	$3 \leq N \leq 8$	$N = 3$	$N = 4$	$N = 5$	$N = 6$	$N = 7$	$N = 8$
6	1	1	1	1	1	1	1
10	1	1	0	0	0	0	0
12	3	0	3	2	2	2	2
14	2	2	0	1	0	0	0
16	2	0	0	1	2	1	1
18	10	5	6	3	3	4	3
20	6	0	0	4	3	3	4
22	14	11	0	3	3	1	1
24	19	0	17	4	6	9	7
26	36	27	0	5	6	3	5
28	20	0	0	13	10	10	10
30	122	78	42	10	5	10	6
32	29	0	0	10	17	11	14
34	256	234	0	12	6	7	9
36	173	0	118	41	28	30	25
38	821	778	0	29	25	15	15
40	77	0	0	24	30	30	35
42	3298	2831	391	49	48	33	21
44	187	0	0	122	42	59	54
46	11288	11122	0	91	70	35	42
48	1526	0	1301	109	100	100	77
total	17891	15090	1879	534	407	364	332

References

1. N.Morikawa, *Research project: Toward Galois theory of protein-like objects*, 2003 (manuscript).

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