

Algebra of affine closed trajectories of
triangles

- Introduction to Hetero numbers (1) -

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This talk is about “algebra of affine closed trajectories of triangles” and is an introduction to the 2-dim'l hetero numbers.

Outline of the talk

- Introduction and motivation
- Lattices, cones, and roofs
- Surface decomposition by conjugate roof
- Algebra of roofs and hetero numbers

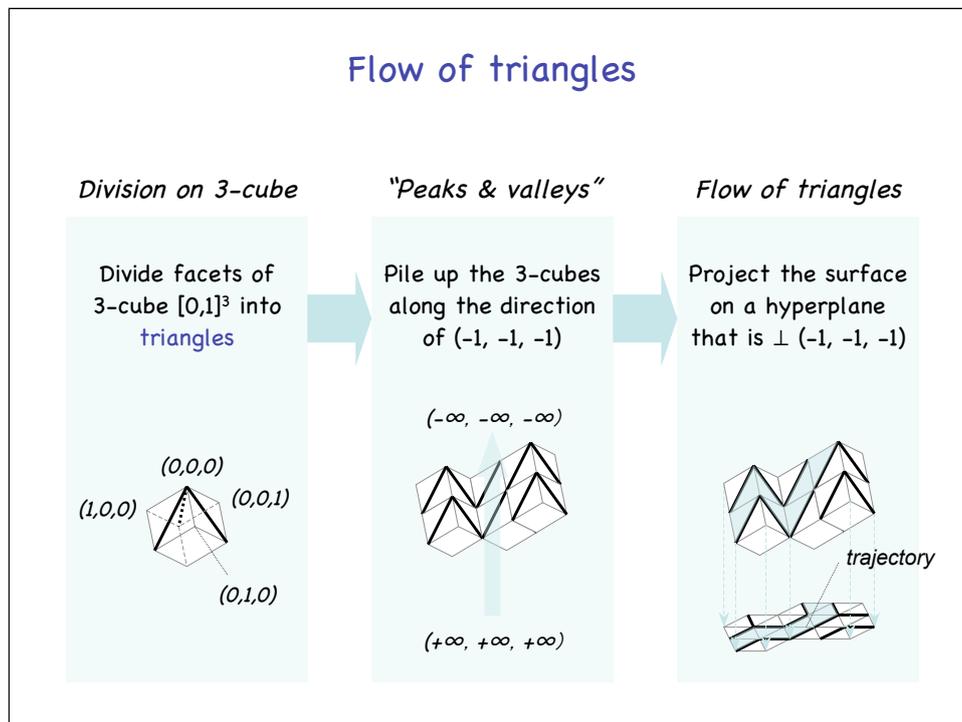
This is the outline of the talk.

First, I'm going to make a brief introduction and give a motivation.

Next, I'll talk about the lattices, cones, and roofs.

Then, I'll talk about the surface decomposition by conjugate roof.

Finally, I'll describe the algebra of roofs and give the definition of 2-dim'l hetero numbers.



The slide shows how flows of triangles are obtained.

First, divide facets of 3-cube $[0,1]^3$ into triangles. As shown on the left, each facet is divided into two triangles.

Next, pile up the 3-cubes along the direction of $(-1, -1, -1)$. That is, they are piled up from $(+\infty, +\infty, +\infty)$ toward $(-\infty, -\infty, -\infty)$.

And we obtain “peaks and valleys” of 3-cubes.

Note that the division of facets makes up a division of the surface of the “peaks and valleys.”

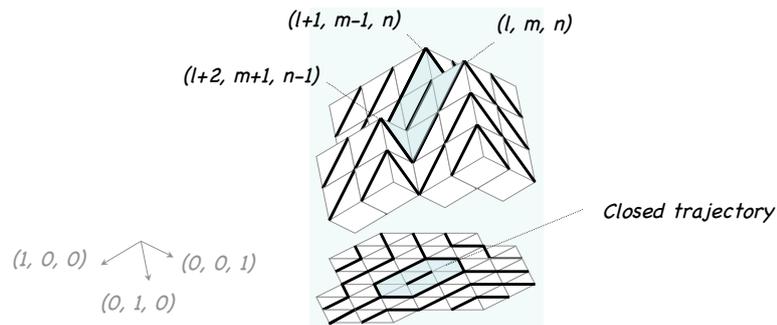
Finally, project the surface on a hyperplane that is perpendicular to vector $(-1, -1, -1)$.

Then, as shown on the right, we obtain a flow of triangles which is induced by the division of the surface.

For example, the blue slant triangles on the “peaks and valleys” specify the blue trajectory of flat triangles on the hyperplane.

Example: affine closed trajectory

- “Peaks and valleys” defined by three peaks, (l, m, n) , $(l+1, m-1, n)$, and $(l+2, m+1, n-1) \in \mathbb{Z}^3$, specifies a closed trajectory of length 10.



[NOTE] *Affine* closed trajectory := closed trajectory specified by a single set of “peaks and valleys”

This is an example of affine closed trajectories of triangles.

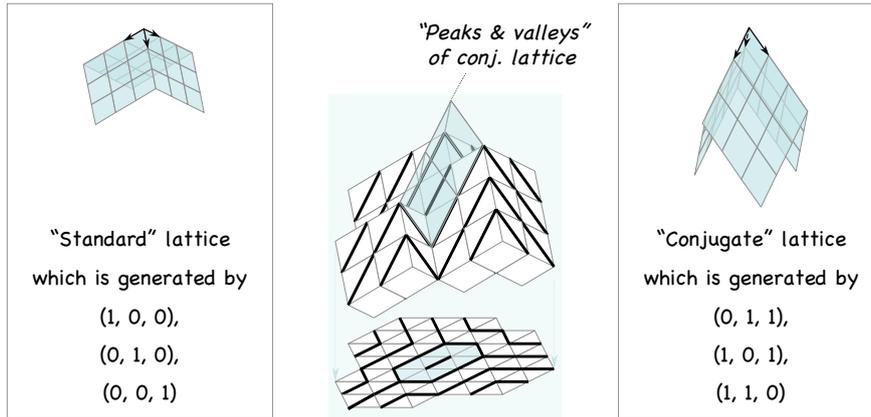
An affine closed trajectory is a closed trajectory specified by a single set of “peaks and valleys.”

And the “peaks and valleys” defined by three peaks, ...

As shown in the figure, the blue slant triangles induce a closed trajectory of flat triangles on the hyperplane.

Motivation of the talk

- One could specify any affine closed trajectory by “peaks and valleys” of the “conjugate” lattice.
- As a result, we could describe “fusion and fission” of closed trajectories as **addition** of the corresponding “conj. peaks & valleys”



This slide shows the motivation of the talk.

That is, as shown in the middle figure, one could specify any affine ...

As a result, we could describe “fusion and fission” of ...

Shown on the left is the “standard” lattice which is generated by $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$.

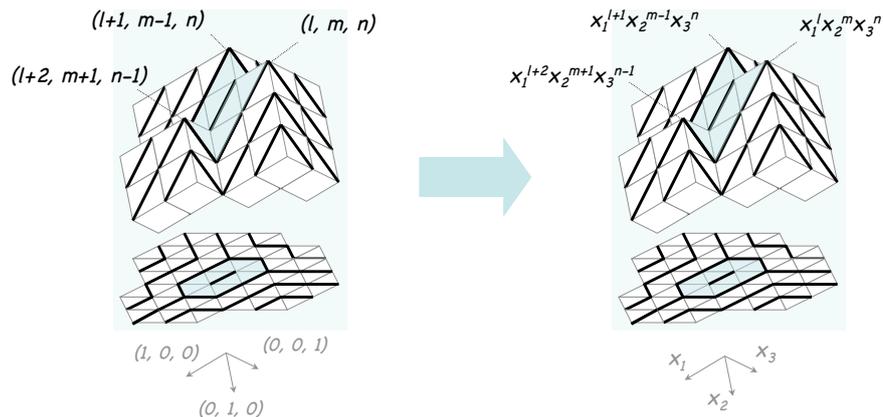
And we have used the lattice to define flow of triangles in the previous slide.

On the other hand, as shown on the right, “conjugate” lattice is the lattice which is generated by $(0, 1, 1)$, $(1, 0, 1)$, and $(1, 1, 0)$.

And, as shown in the middle figure, only the slant triangles which induce the closed trajectory on the hyperplane are just covered by “peaks and valleys” of the conjugate lattice which is colored blue.

Notation: monomial representation

- We denote point $(l, m, n) \in \mathbb{R}^3$ by monomial $x_1^l x_2^m x_3^n$ of three indeterminates $x_1, x_2,$ and x_3 :



Before giving the definition of the conjugate cone associated with an affine closed trajectory, let me explain the notation used.

In this talk, we denote point (l, m, n) ...

For example, the “peaks and valleys” on the left is described as shown on the right.

And, vectors $(1, 0, 0)$, $(0, 0, 1)$, and $(0, 1, 0)$ are denoted by $x_1, x_2,$ and x_3 respectively.

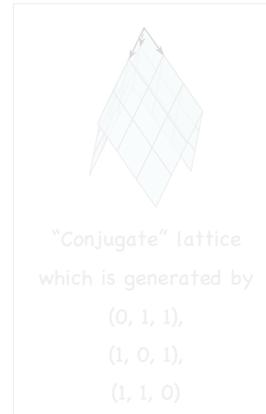
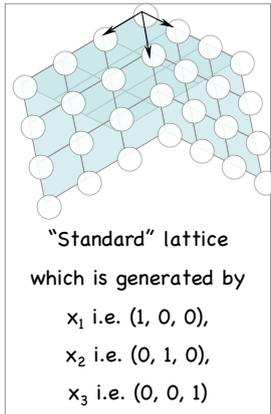
Outline of the talk

- Introduction and motivation
- **Lattices, cones, and roofs**
- Surface decomposition by conjugate roof
- Algebra of roofs and hetero numbers

Now, let me talk about lattices, cones, and roofs.

Standard lattice

- [Def'n] The *standard lattice* L_3 is the lattice which is generated by x_1 , x_2 , and x_3
- The standard lattice L_3 is used to specify a **flow** of triangles



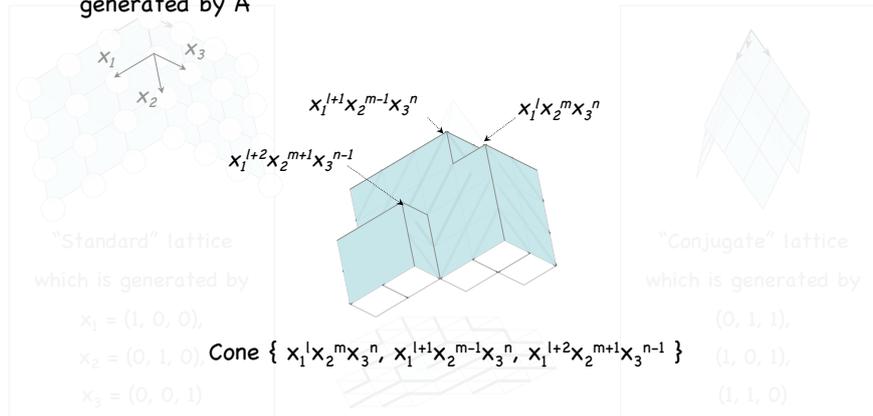
Let's start with the standard lattice.

As shown in the figure, the standard lattice L_3 is the lattice which is ...

And the standard lattice L_3 is used to ...

Standard cone ("tangent" cone)

- [Def'n] For $A = \{ x_1^{l_i} x_2^{m_i} x_3^{n_i} \} \subset L_3$,
 $\text{Cone } A := \{ (l, m, n) \in \mathbf{R}^3 : l \geq l_i, m \geq m_i, n \geq n_i \text{ for some } x_1^{l_i} x_2^{m_i} x_3^{n_i} \in A \}$
- Cone A specifies "peaks and valleys" of the standard lattice generated by A



Standard cone is a cone of the standard lattice defined as follows.

That is, for a subset A of standard lattice L_3 , Cone A is a set of points (l, m, n) in \mathbf{R}^3 which satisfy ...

Cone A specifies "peaks and valleys" of the standard lattice ...

For example, as shown below, the "peaks and valleys" of the previous example is Cone $\{ x_1^l x_2^m x_3^n, x_1^{l+1} x_2^{m-1} x_3^n, x_1^{l+2} x_2^{m+1} x_3^{n-1} \}$.

Since standard cones specify the "gradient" of slant triangles, we could call them "tangent" cone.

Conjugate lattice

- [Def'n] The *conjugate lattice* L_3^* is the lattice which is generated by x_2x_3 , x_1x_3 , and x_1x_2
- The conjugate lattice L_3^* is used to specify the "boundary" of a trajectory



Conjugate lattice is defines as follows.

That is, as shown on the right , the conjugate lattice L_3^* is the lattice ...

And the conjugate lattice L_3^* is used to ... as we will see later.

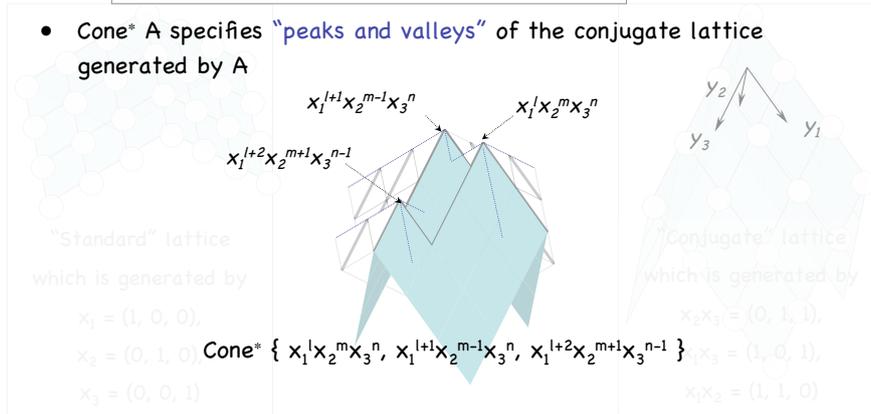
Note that, as shown in the left, the conjugate lattice is "sparse" and it covers only the white points of the standard lattice.

Conjugate cone ("cotangent" cone)

- [Def'n] For $A = \{ x_1^{l_i} x_2^{m_i} x_3^{n_i} \} \subset L_3$,
 $\text{Cone}^* A := \{ (l, m, n) \in \mathbb{R}^3 : -l+m+n \geq -l_i+m_i+n_i, l-m+n \geq l_i-m_i+n_i, l+m-n \geq l_i+m_i-n_i, \text{ for some } x_1^{l_i} x_2^{m_i} x_3^{n_i} \in A \}$

Note that $x_1^l x_2^m x_3^n = y_1^{(-l+m+n)/2} y_2^{(l-m+n)/2} y_3^{(l+m-n)/2}$.

- $\text{Cone}^* A$ specifies "peaks and valleys" of the conjugate lattice generated by A



Conjugate cone is a cone of the conjugate lattice defined as follows.

That is, for subset A of standard lattice L_3 , $\text{Cone}^* A$ is a set of points (l, m, n) in \mathbb{R}^3 which satisfy ...

Note that point (l, m, n) of the standard lattice corresponds to point $((-l+m+n)/2, (l-m+n)/2, (l+m-n)/2)$ of the conjugate lattice.

As shown below, $\text{Cone}^* A$ specifies "peaks and valleys" of the conjugate lattice ...

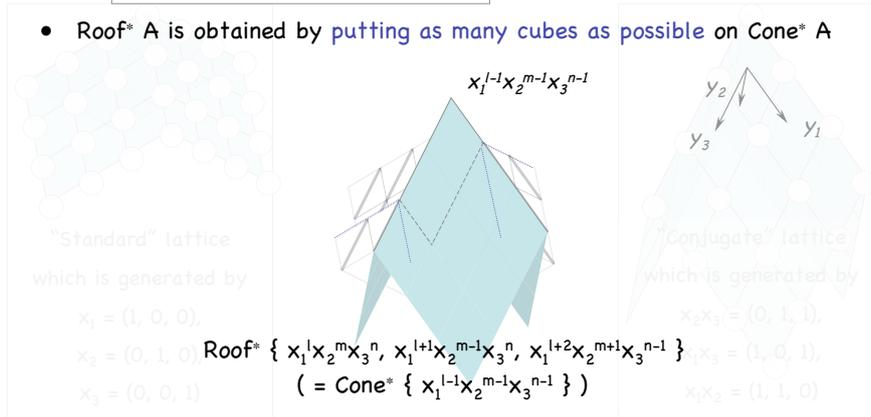
Since conjugate cones specify the "boundary" of slant triangles, we could call them "cotangent" cone.

Conjugate roof

- [Def'n] For $A \subset L_3$,
 $\text{Roof}^* A := \{ (m+n, l+n, l+m) \in \mathbb{R}^3 : y_1^{l+n} y_2^m y_3^n, y_1^l y_2^{m+n} y_3^n, y_1^l y_2^m y_3^{n+N} \in \text{Cone}^* A \text{ for some } N > 0 \}$

Note that $x_1^{m+n} x_2^{l+n} x_3^{l+m} = y_1^l y_2^m y_3^n$.

- $\text{Roof}^* A$ is obtained by putting as many cubes as possible on $\text{Cone}^* A$



To specify affine closed trajectories, a particular type of conjugate cones, called “conjugate roof”, are used.

For subset A of standard lattice L_3 , $\text{Roof}^* A$ is a set of points $(m+n, l+n, l+m)$ in \mathbb{R}^3 which satisfy ...

Note that point $(m+n, l+n, l+m)$ of the standard lattice corresponds to point (l, m, n) of the conjugate lattice.

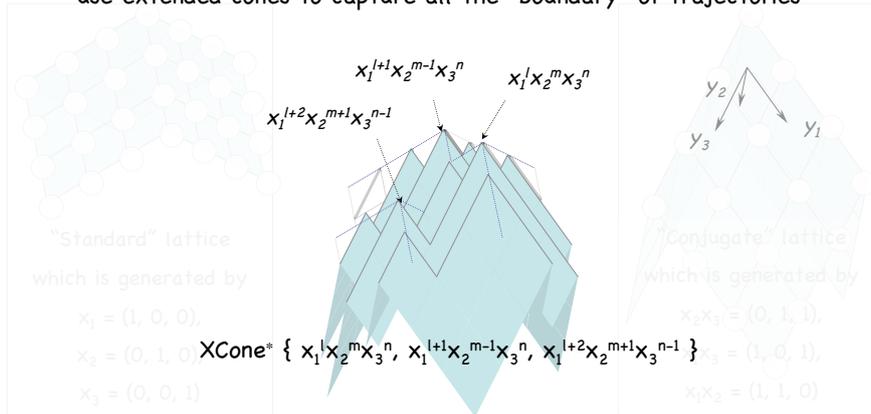
As shown below, $\text{Roof}^* A$ is obtained by putting as ...

In this case, two cubes are put on $\text{Cone}^* A$ to obtain $\text{Roof}^* A$ which is equal to $\text{Cone}^* \{ x_1^{l-1} x_2^{m-1} x_3^{n-1} \}$.

Extended conjugate cone

- [Def'n] For $A \subset L_3$,

$$\text{XCone}^* A := \text{Cone}^* A \cup \text{Cone}^* x_1 A \cup \text{Cone}^* x_2 A \cup \text{Cone}^* x_3 A,$$
 where $x_i A = \{ x_i a \mid a \in A \}$
- Since the conjugate lattice L_3^* is "sparse" ($L_3 \sim L_3^* + x_1 L_3^*$), one should use extended cones to capture all the "boundary" of trajectories



Finally, let me give the definition of extended conjugate cone, which is mainly used in the 3-dim'l case.

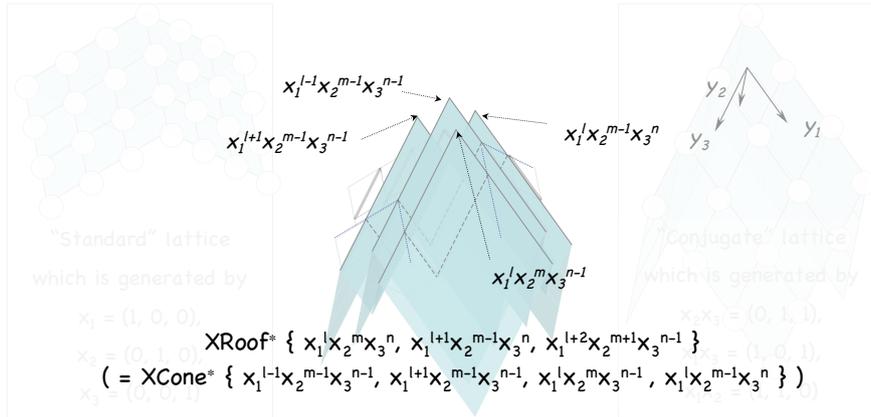
For subset A of standard lattice L_3 , $\text{XCone}^* A$ is a union of four conjugate cones $\text{Cone}^* A$, $\text{Cone}^* x_1 A$, $\text{Cone}^* x_2 A$, and $\text{Cone}^* x_3 A$.

Since the conjugate lattice is sparse ...

As shown below, the extended conjugate cone has more peaks than the normal conjugate cone.

Extended conjugate roof

- [Def'n] For $A \subset L_3$,
 $\text{XRoof}^* A :=$ the conjugate roof of $\text{XCone}^* A$
- $\text{XRoof}^* A$ is obtained by putting as many cubes as possible on $\text{XCone}^* A$



Extended conjugate roof is defined in the same way as normal conjugate cones.

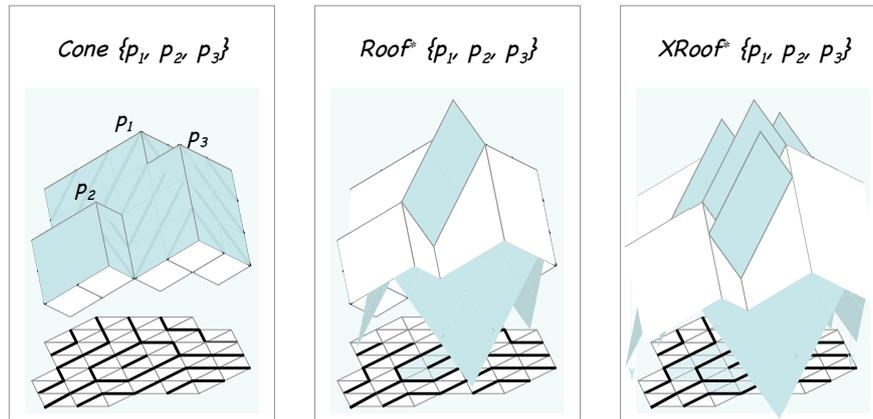
For subset A of standard lattice L_3 , $\text{XRoof}^* A$ is ...

As shown below, $\text{XRoof}^* A$ is obtained by putting as ...

In this case, eight cubes are put on $\text{XCone}^* A$ to obtain $\text{XRoof}^* A$ which is equal to $\text{XCone}^* \{ x_1^{l-1} x_2^{m-1} x_3^{n-1}, \dots \}$.

Summary of lattices, cones, and roofs

- Standard cone **Cone A** is defined to specify the "gradient" of triangles.
 ==> Cone A is used to specify **a flow of tetrahedrons**
- Conjugate roof **Roof* A** is defined to specify the "boundary" of a traj.
 ==> Roof* A (or XRoof* A) is used to specify **a trajectory**



This is the summary of lattices, cones, and roofs.

Firstly, standard cone Cone A is defined to ...

As shown on the left, Cone A is used to ...

Secondly, conjugate roof Roof* A is defined to ...

As we will see later, Roof* A (or XRoof* A) is used to ...

Shown in the middle is the conjugate roof associated with Cone $\{p_1, p_2, p_3\}$ of the left figure.

And shown on the right is the extended conjugate roof associated with the cone.

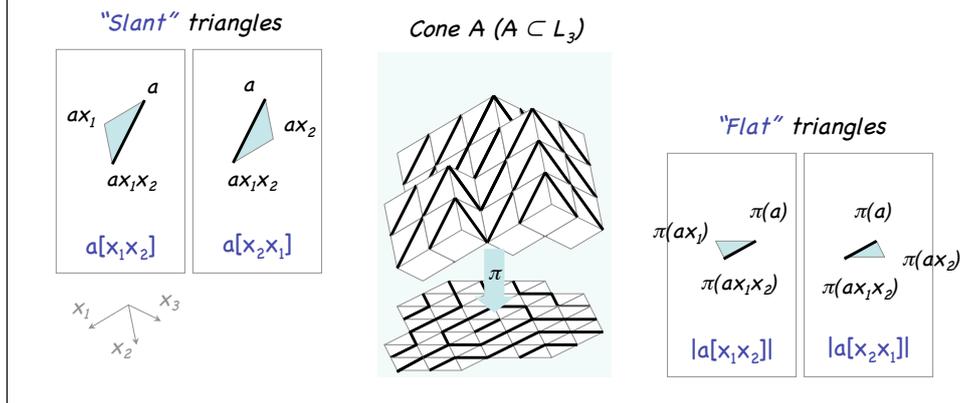
Outline of the talk

- Introduction and motivation
- Lattices, cones, and roofs
- **Surface decomposition by conjugate roof**
- Algebra of roofs and hetero numbers

Next, let me talk about surface decomposition by conjugate roof.

Notation : triangles on Cone A

- We denote the **triangle** of vertices $a (=x_1^m x_2^n x_3^p)$, ax_i , $ax_i x_j$ by $a[x_i x_j]$
- We denote the **projection** of $a[x_i x_j]$ on the hyperplane by $|a[x_i x_j]|$



Before giving the definition of the decomposition, let me explain the notation used.

As shown on the left, we denote the triangle of ...

For example, vertices a , ax_1 , and ax_1x_2 form slant triangle $a[x_1x_2]$.

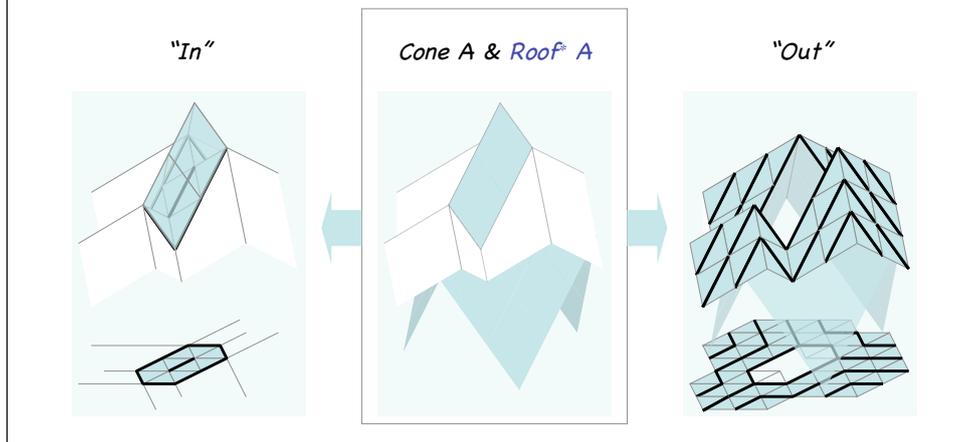
And, as shown on the right, we denote the projection of ...

For example, vertices $\pi(a)$, $\pi(ax_1)$, and $\pi(ax_1x_2)$ form flat triangle $|a[x_1x_2]|$,

where π is the projection of "peaks and valleys" onto the hyperplane.

Surface decomposition by Roof* A

- Slant triangles on Cone A ($A \subset L_3$) are classified into three kinds:
 - $\text{In}(\text{Cone } A, \text{Roof}^* A) :=$ the slant triangles **under** Roof* A
 - $\text{Bd}(\text{Cone } A, \text{Roof}^* A) :=$ the slant triangles **partially covered** by Roof* A
 - $\text{Out}(\text{Cone } A, \text{Roof}^* A) :=$ the slant triangles **above** Roof* A



This slide shows the surface decomposition by conjugate roof Roof* A.

Slant triangles on standard cone Cone A are classified into three kinds:

$\text{In}(\text{Cone } A, \text{Roof}^* A)$ is the set of all the slant triangles under Roof* A,

$\text{Bd}(\text{Cone } A, \text{Roof}^* A)$ is the set of all the slant triangles partially covered by Roof* A,

And $\text{Out}(\text{Cone } A, \text{Roof}^* A)$ is the set of all the slant triangles above Roof* A.

Shown below is the decomposition of slant triangles on Cone A of the middle figure, where the conjugate roof Roof* A is colored blue.

Shown on the left are the slant triangles under Roof* A and their image on the hyperplane.

Shown on the right are the slant triangles above Roof* A and their image on the hyperplane.

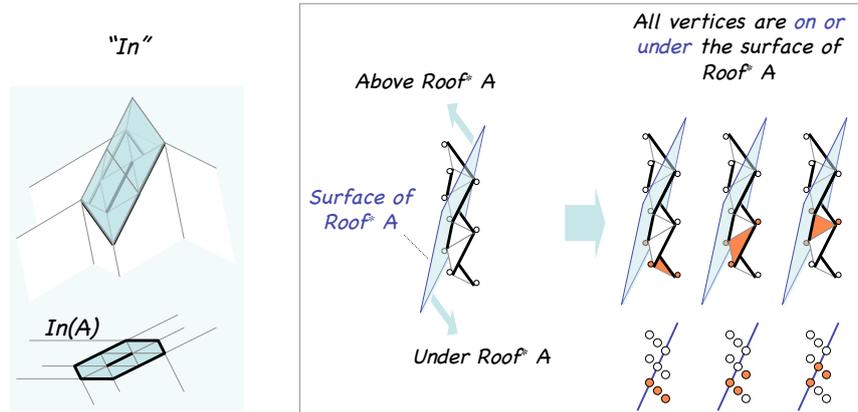
In this example, there are no slant triangles partially covered by Roof* A.

Surface decomposition (cont'd)

- Slant triangles on Cone A ($A \subset L_3$) are classified into three kinds:
 - $\text{In}(\text{Cone } A, \text{Roof}^* A) :=$ the slant triangles under $\text{Roof}^* A$
- We denote the projection on the hyperplane by $\text{In}(A)$:

$$\text{In}(A) := \{ |a[x_i, x_j]| \mid a[x_i, x_j] \in \text{In}(\text{Cone } A, \text{Roof}^* A) \}$$

A



Firstly, let's examine more details about the "In" triangles.

$\text{In}(\text{Cone } A, \text{Roof}^* A)$ is the set of all the slant triangles under $\text{Roof}^* A$.

And we denote the projection on the hyperplane by $\text{In}(A)$.

That is, $\text{In}(A)$ is the set of all the flat triangles whose corresponding slant triangle belongs to $\text{In}(\text{Cone } A, \text{Roof}^* A)$, as shown on the left.

Shown in the middle figure is the relative position of slant triangles and the conjugate roof $\text{Roof}^* A$.

The blue plane denotes a surface of $\text{Roof}^* A$ and arrows indicate the direction of "above" and "under" the surface.

And, as shown on the right, $\text{In}(\text{Cone } A, \text{Roof}^* A)$ consists of the slant triangles whose vertices are on or under the surface of $\text{Roof}^* A$.

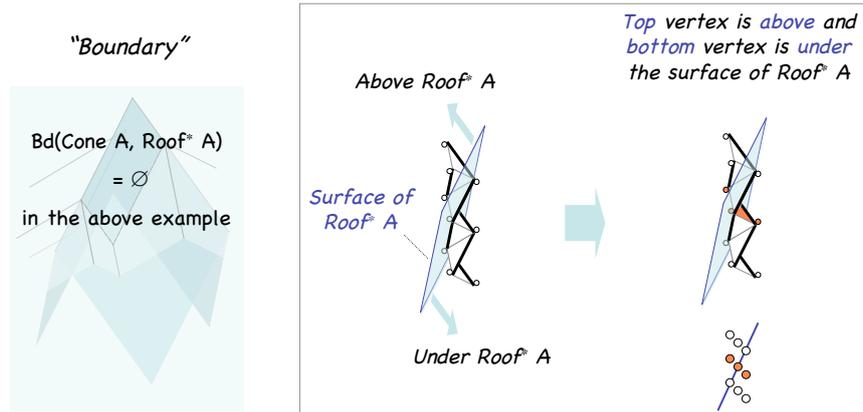
For example, the left figure (of the figure) shows the bottom triangle (colored red), whose top vertex is on the surface and the others are under the surface.

The middle figure shows the second triangle from the bottom, whose second vertex is on the surface and the others are under the surface.

The right figure shows the third triangle from the bottom, whose second vertex is on the surface and the others are under the surface.

Surface decomposition (cont'd)

- Slant triangles on Cone A ($A \subset L_3$) are classified into three kinds:
 - $\text{In}(\text{Cone } A, \text{Roof}^* A) :=$ the slant triangles under $\text{Roof}^* A$
 - $\text{Bd}(\text{Cone } A, \text{Roof}^* A) :=$ the slant triangles **partially covered** by $\text{Roof}^* A$
 - $\text{Out}(\text{Cone } A, \text{Roof}^* A) :=$ the slant triangles above $\text{Roof}^* A$



Secondly, let's examine more details about the "Boundary" triangles.

$\text{Bd}(\text{Cone } A, \text{Roof}^* A)$ is the set of all the slant triangles partially covered by $\text{Roof}^* A$.

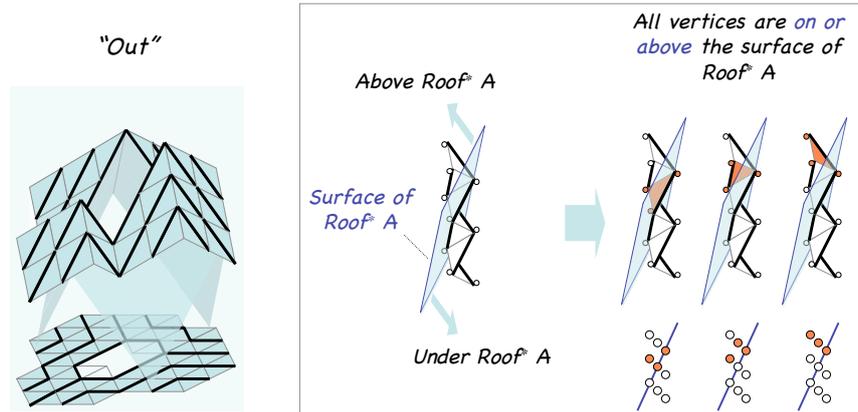
And $\text{Bd}(\text{Cone } A, \text{Roof}^* A)$ is empty in the above example.

As shown on the right, $\text{Bd}(\text{Cone } A, \text{Roof}^* A)$ consists of the slant triangles whose top vertex is above and bottom vertex is under the surface of $\text{Roof}^* A$.

The figure shows the fourth triangle (colored red) from the bottom, whose top vertex is above the surface and bottom vertex is under the surface.

Surface decomposition (cont'd)

- Slant triangles on Cone A ($A \subset L_3$) are classified into three kinds:
 - $\text{In}(\text{Cone } A, \text{Roof}^* A) :=$ the slant triangles under $\text{Roof}^* A$
 - $\text{Bd}(\text{Cone } A, \text{Roof}^* A) :=$ the slant triangles partially covered by $\text{Roof}^* A$
 - $\text{Out}(\text{Cone } A, \text{Roof}^* A) :=$ the slant triangles **above** $\text{Roof}^* A$



Finally, let's examine more details about the "Out" triangles.

$\text{Out}(\text{Cone } A, \text{Roof}^* A)$ is the set of all the slant triangles above $\text{Roof}^* A$.

As shown on the right, $\text{Out}(\text{Cone } A, \text{Roof}^* A)$ consists of the slant triangles whose vertices are on or above the surface of $\text{Roof}^* A$.

For example, the left figure (of the figure) shows the third triangle (colored red) from the top, whose top and bottom vertices are on the surface and the other is above the surface.

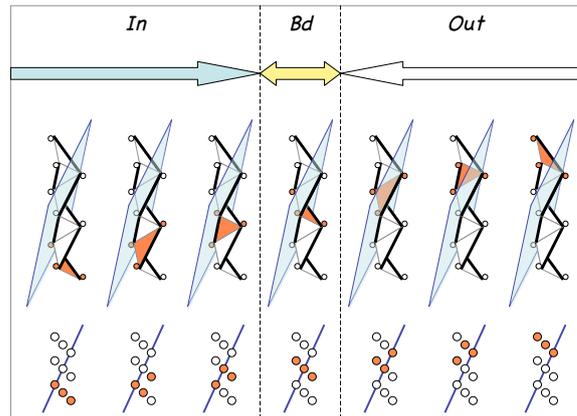
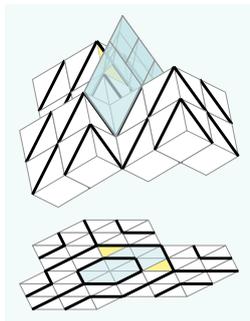
The middle figure shows the second triangle from the top, whose second vertex is on the surface and the others are above the surface.

The right figure shows the top triangle, whose bottom vertex is on the surface and the others are above the surface.

Surface decomposition (summary)

- Slant triangles on Cone A ($A \subset L_3$) are classified into three kinds:
 - $\text{In}(\text{Cone } A, \text{Roof}^* A) :=$ the slant triangles **under** $\text{Roof}^* A$
 - $\text{Bd}(\text{Cone } A, \text{Roof}^* A) :=$ the slant triangles **partially covered** by $\text{Roof}^* A$
 - $\text{Out}(\text{Cone } A, \text{Roof}^* A) :=$ the slant triangles **above** $\text{Roof}^* A$

Another example



This is the summary of the surface decomposition.

As we see, slant triangles on Cone A ...

Shown on the left is another example, where blue, yellow, and white flat triangles indicate the area of “In”, “Boundary”, and “Out.”

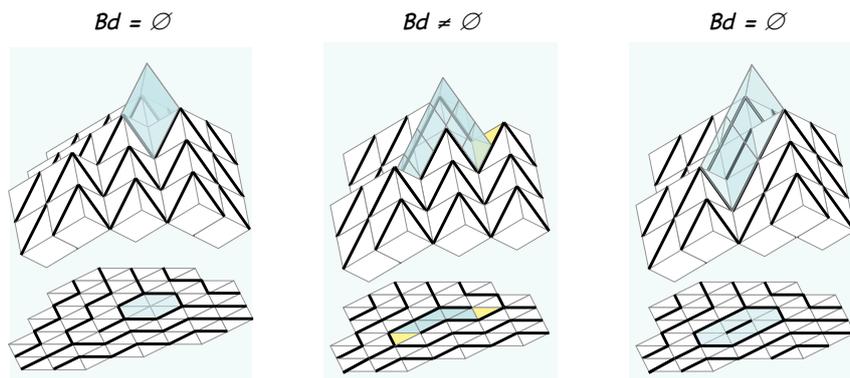
Shown on the right is a schematic diagram of the relative position of slant triangles and the conjugate roof.

Slant triangles belong to “In” until the top vertex crosses the surface of the conjugate roof.

Then, it belongs to “Boundary”. All the triangles above the “Boundary” triangle belong to “Out.”

Decomposition & Trajectory

- Given $A \subset L_3$. If $Bd(\text{Cone } A, \text{Roof}^* A) = \emptyset$, then no trajectory defined by Cone A crosses the boundary of "In" and "Out". In particular,
 - $\text{In}(\text{Cone } A, \text{Roof}^* A) \Leftrightarrow$ all the closed trajectories
 - $\text{Out}(\text{Cone } A, \text{Roof}^* A) \Leftrightarrow$ all the open trajectories



Now let me show the relation between the decomposition and trajectories of triangles.

Given subset A of standard lattice L_3 . If $Bd(\text{Cone } A, \text{Roof}^* A)$ is empty, then ...

In particular,

$\text{In}(\text{Cone } A, \text{Roof}^* A)$ consists of all the closed trajectories, and

$\text{Out}(\text{Cone } A, \text{Roof}^* A)$ consists of all the open trajectories.

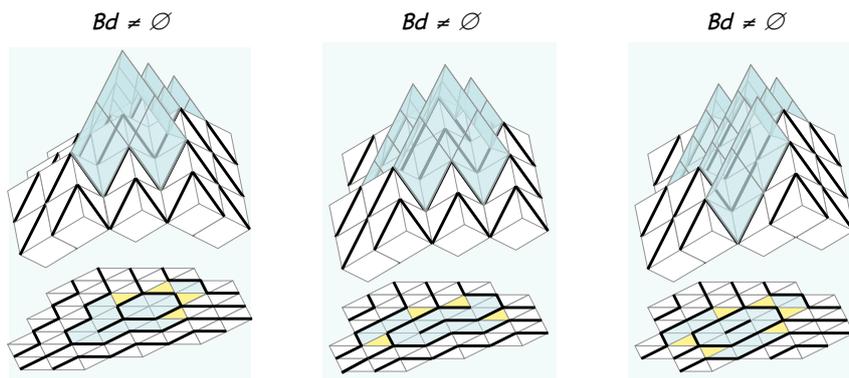
In the left example, there is no "Boundary" triangle and "In" triangles form the only closed trajectory of length six.

In the middle example, there are two "Boundary" triangles (colored yellow) and "In" triangles are part of a open trajectory.

In the right example, there is no "Boundary" triangle and "In" triangles form the only closed trajectory of length ten.

Decomposition & Trajectory (cont'd)

- Extended conjugate roof $X\text{Roof}^* A$ gives a "larger neighborhood":
 - $\text{In}(\text{Cone } A, X\text{Roof}^* A)$ includes all the closed trajectories
 - Usually, $\text{Bd}(\text{Cone } A, X\text{Roof}^* A) \neq \emptyset$
- $X\text{Roof}^* A$ is mainly used in the 3-dim'l case to capture all the closed trajectories which visit *at least one* of the peaks of Cone A



This slide shows the decomposition by extended conjugate roof.

We could use extended conjugate roofs, instead of normal conjugate roofs.

And extended conjugate roof $X\text{Roof}^* A$ gives ...

$X\text{Roof}^* A$ is mainly used in ...

Shown below are the same examples of the previous slide.

In the left example, "In" triangles form the only closed trajectory and part of two open trajectories.

In the middle example, "In" triangles form part of two open trajectory.

In the right example, "In" triangles form the only closed trajectory and part of three open trajectories.

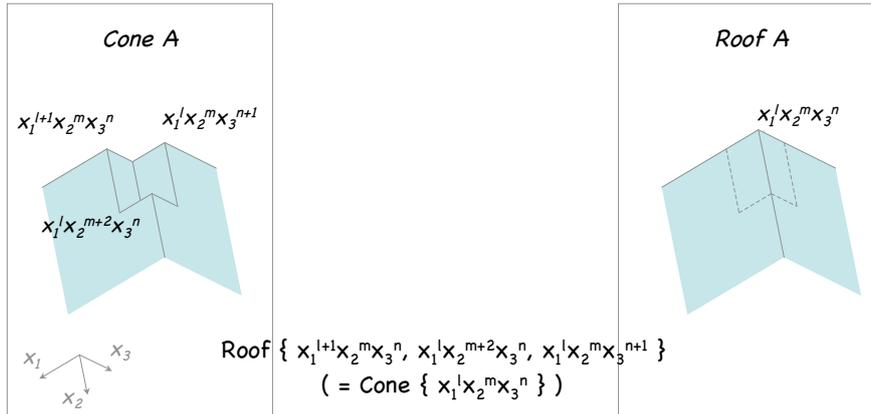
Outline of the talk

- Introduction and motivation
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Finally, let me talk about algebra of roofs and hetero numbers.

Standard roof

- [Def'n] For $A \subset L_3$,
 $\text{Roof } A := \{ (l, m, n) \in \mathbb{R}^3 : (l + N, m, n), (l, m + N, n), (l, m, n + N) \in \text{Cone } A \text{ for some } N > 0 \}$
- Roof A is obtained by putting as many cubes as possible on Cone A



Firstly, let's consider another type of roof, that is, standard roof.

And here is the definition.

For subset A of standard lattice L_3 , Roof A is a set of points (l, m, n) in \mathbb{R}^3 which satisfy ...

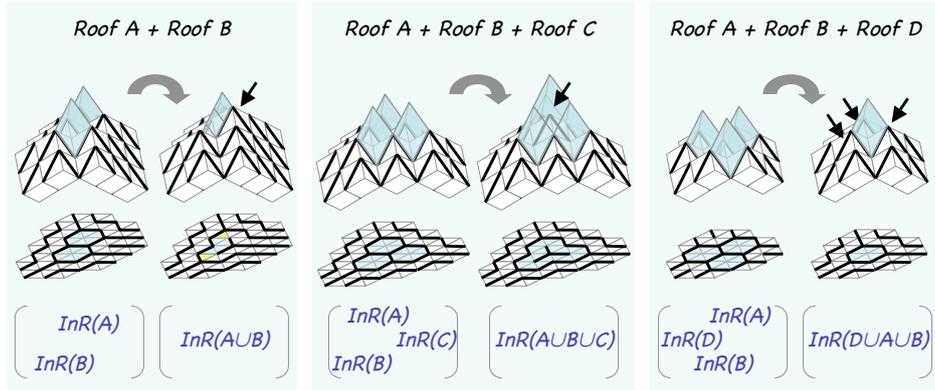
As shown below, Roof* A is obtained by putting as ...

Shown on the left is a standard cone generated by three peaks, $x_1^{l+1}x_2^mx_3^n$, $x_1^lx_2^{m+2}x_3^n$, and $x_1^lx_2^mx_3^{n+1}$.

And, by putting two cubes on the cone, we obtain Roof $\{ \dots \}$, shown on the right, which is equal to Cone $\{ x_1^lx_2^mx_3^n \}$.

Addition of roofs

- [Def'n] For A and $B \subset L_3$,
 $\text{Roof } A + \text{Roof } B := \text{Roof } A \cup B$
- In general, $\text{InR}(A) \cup \text{InR}(B) \neq \text{InR}(A \cup B)$,
 where $\text{InR}(X) := \{ |a[x_i, x_j]| \mid a[x_i, x_j] \in \text{In}(\text{Roof } X, \text{Roof}^* X) \}$



* Recall that $\text{In}(X) := \{ |a[x_i, x_j]| \mid a[x_i, x_j] \in \text{In}(\text{Cone } X, \text{Roof}^* X) \}$.

Then, addition of roofs is defined as follows.

For subset A and B of standard lattice L_3 , $\text{Roof } A + \text{Roof } B$ is ...

As shown below, in general, $\text{InR}(A) \cup \text{InR}(B)$ is not equal to $\text{InR}(A \cup B)$, where $\text{InR}(X)$ is the projection of $\text{In}(\text{Roof } X, \text{Roof}^* X)$ on the hyperplane.

Shown on the left is addition of two roofs $\text{Roof } A$ and $\text{Roof } B$, where closed trajectories $\text{In}(A)$ and $\text{In}(B)$ break upon addition.

The black arrow indicates the cube put on to obtain the roof.

Shown in the middle is addition of three roofs $\text{Roof } A$, $\text{Roof } B$ and $\text{Cone } C$, where closed trajectories $\text{In}(A)$, $\text{In}(B)$, and $\text{In}(C)$ fuse into a single closed trajectory $\text{In}(A \cup B \cup C)$ upon addition.

Shown on the right is addition of another triplet of roofs $\text{Roof } A$, $\text{Roof } B$ and $\text{Cone } D$, where closed trajectories $\text{In}(A)$, $\text{In}(B)$, and $\text{In}(C)$ collapse into a shorter closed trajectory $\text{In}(A \cup B \cup D)$ upon addition.

Addition of conjugate roofs is also defined similarly.

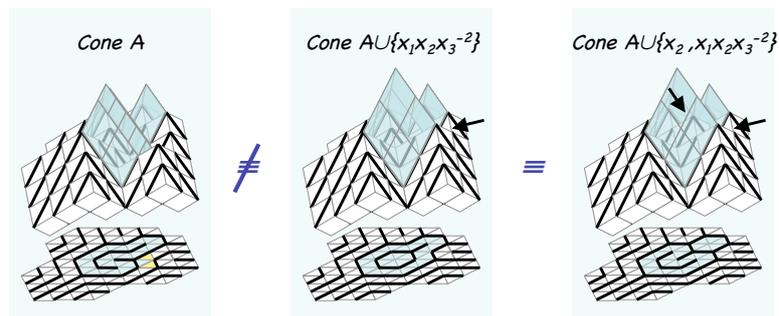
Equivalence relation on cones

- [Def'n] For A and $B \subset L_3$, we introduce an *equiv. relation* on cones:

$$\text{Cone } A \equiv \text{Cone } B \iff \begin{cases} \text{Roof } A = \text{Roof } B, \\ \text{Roof}^* g(A) = \text{Roof}^* g(B) \end{cases}$$

where $g(X)$ denotes the peaks of Cone X . The equivalence class of Cone A is denoted by $[\text{Cone } A]_R$

- Then, $\text{In}(A) = \text{In}(B)$ if $[\text{Cone } A]_R = [\text{Cone } B]_R$



where $A = \{ x_1, x_3, x_1x_2^2x_3^{-2}, x_1^{-1}x_2x_3, x_1^{-1}x_2^3 \}$

Next, let's consider an equivalence relation on standard cones.

For subset A and B of standard lattice L_3 , ...

Then, as shown below, $\text{In}(A)$ is equal to $\text{In}(B)$ if ...

Shown on the left is the flow defined by Cone A , where A is given below.

Shown in the middle is the flow defined by another cone which is obtained by putting one cube (indicated by an arrow) on Cone A .

By the construction, both cones are associated with the same standard roof.

But their corresponding conjugate roofs, colored blue, are different from each other, as you see.

Therefore, they are not equivalent and their closed trajectories sweep different areas on the hyperplane.

On the other hand, shown on the right is the flow defined by another cone which is obtained by putting one more cube.

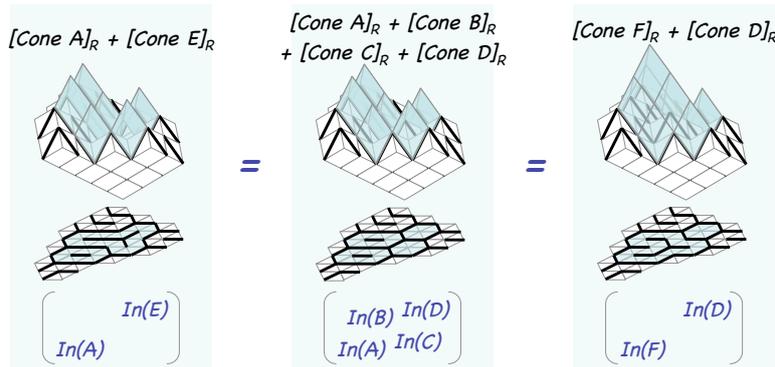
By the construction, this cone is also associated with the same standard roof.

Moreover, as you see, the right two cones share the same conjugate

Addition of $[\text{Cone } A]_R$

- [Def'n] For A and $B \subset L_3$,
 $[\text{Cone } A]_R + [\text{Cone } B]_R := [\text{Cone } A \cup B]_R$
- If $g(A) \cup g(B) = g(C)$, then $[\text{Cone } A]_R + [\text{Cone } B]_R = [\text{Cone } C]_R \Leftrightarrow$

$$\begin{cases} \text{Roof } A + \text{Roof } B = \text{Roof } C, \\ \text{Roof}^* g(A) + \text{Roof}^* g(B) = \text{Roof}^* g(C) \end{cases}$$



Now, addition of the equivalence classes is defined as follows.

For subset A and B of standard lattice L_3 , ...

Note that, if $g(A) \cup g(B) = g(C)$, then $[\text{Cone } A]_R + [\text{Cone } B]_R = [\text{Cone } C]_R$ if and only if ...

Shown on the left is addition of two classes $[\text{Cone } A]_R + [\text{Cone } E]_R$, where the blue closed trajectories below show the shape of $\text{In}(A)$ and $\text{In}(E)$.

Shown in the middle is addition of four classes $[\text{Cone } A]_R + [\text{Cone } B]_R + [\text{Cone } C]_R + [\text{Cone } D]_R$, where the blue trajectories below show the shape of $\text{In}(A)$, $\text{In}(B)$, $\text{In}(C)$, and $\text{In}(D)$.

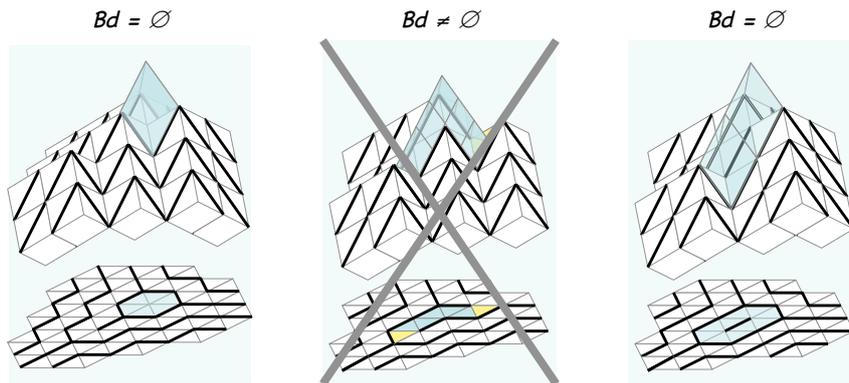
And shown on the right is addition of two classes $[\text{Cone } F]_R + [\text{Cone } D]_R$, where the blue trajectories below show the shape of $\text{In}(D)$ and $\text{In}(F)$.

As you see, all of them are associated with the same standard roof and the same conjugate roof.

Thus, they belong to the same equivalent class. In other words, the equivalence class has three different algebraic expressions.

Hetero Numbers

- [Def'n] *Hetero numbers HN* are defined as follows:
 - $HN := \{ [Cone A]_R : A \subset L_3, Bd(Cone A, Roof^* A) = \emptyset \},$
 - $[Cone A]_R + [Cone B]_R := [Cone A \cup B]_R$
- Recall that, if $Bd(Cone A, Roof^* A) = \emptyset$, then
 - $In(Cone A, Roof^* A) \Leftrightarrow$ all the closed trajectories



Now we could give the definition of hetero numbers.

Hetero numbers HN are defined as ...

$HN := \dots$

with addition defined by ...

$[Cone A]_R + [Cone B]_R := \dots$

Recall that ...

Shown below are some examples.

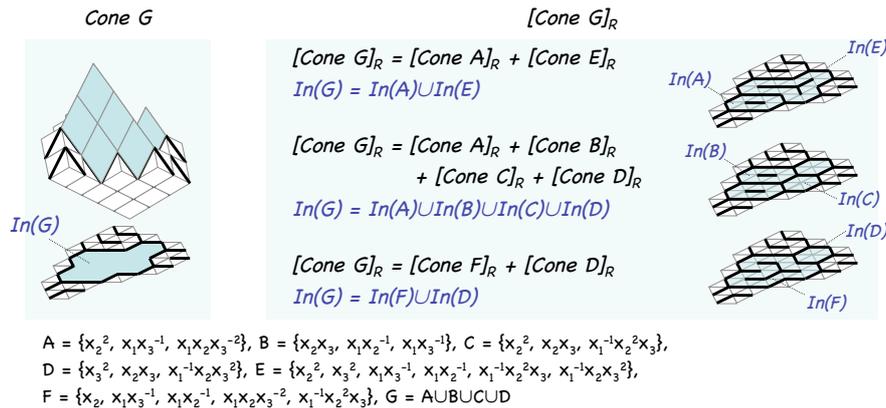
In the left example, there is no “Boundary” triangle and the equivalent class of the cone is a hetero number.

In the middle example, there are two “Boundary” triangles (colored yellow) and the equivalent class of the cone is not a hetero number.

In the right example, there is no “Boundary” triangle and the equivalent class of the cone is a hetero number.

Example: "fusion & fission" of closed traj.s

- For $[Cone A]_R \in \mathbf{HN}$, the equiv. class gives all the "partitions" of $In(A)$ into closed trajectories
- In particular, we could describe "fusion and fission" of closed trajectories as "addition" of the corresponding hetero numbers



Finally, as an example of hetero numbers, let's consider "fusion and fission" of closed trajectories.

For hetero number $[Cone A]_R$, the cones in the equivalent class gives all the ...

In particular, we could describe ..., as shown below.

Shown on the left is Cone G and its associated conjugate roof (colored blue).

They specify the shape of $In(G)$ on the hyperplane. The contents of G and others are given below.

Shown on the right is all the cones in the equivalent class of Cone G. As you see, there are three cones.

The first one is associated with two closed trajectories which have origin in Cone A and Cone E respectively.

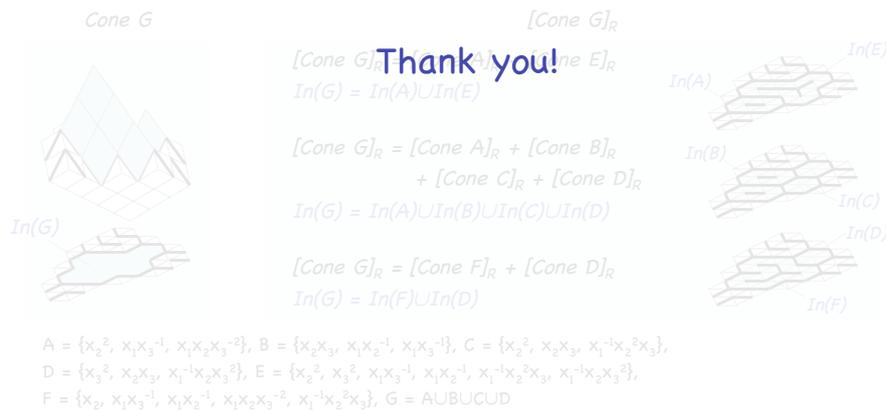
Thus, it corresponds to $[Cone A]_R + [Cone E]_R$.

And the partition of $In(G)$ into two closed trajectories, $In(A)$ and $In(E)$, is expressed as addition $[Cone G]_R = [Cone A]_R + [Cone E]_R$.

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